ON IRREDUCIBLE ALGEBRAS OF CONFORMAL ENDOMORPHISMS OVER A LINEAR ALGEBRAIC GROUP

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ABSTRACT. We study the algebra of conformal endomorphisms $\mathrm{Cend}_n^{G,G}$ of a finitely generated free module M_n over the coordinate Hopf algebra H of a linear algebraic group G. It is shown that a conformal subalgebra of Cend_n acting irreducibly on M_n generates an essential left ideal of $\mathrm{Cend}_n^{G,G}$ if enriched with operators of multiplication on elements of H. In particular, we describe such subalgebras for the case when G is finite.

Introduction. The notion of a conformal algebra was introduced in [1] as a tool for investigation of vertex algebras [2, 3]. From the formal point of view, a conformal algebra is a linear space C over a field $\Bbbk$ ($\operatorname{char}\Bbbk = 0$) endowed with a linear operator $T : C \to C$ and with a family of bilinear operations $(\cdot\ {}_n\ \cdot)$, $n \in \mathbb{Z}_+$ (where $\mathbb{Z}_+$ stands for the set of non-negative integers), satisfying the following axioms:

(C1) for every $a, b \in C$ there exists $N \in \mathbb{Z}_+$ such that $(a\ {}_n\ b) = 0$ for all $n \geq N$;

(C2) $(Ta\ {}_n\ b) = T(a\ {}_n\ b) - (a\ {}_n\ Tb) = \begin{cases} -n(a\ {}_{n-1}\ b), & n > 0 \\ 0, & n = 0 \end{cases}$

One of the most natural examples of a conformal algebra is the Weyl conformal algebra $\mathcal{W} = \Bbbk[T]^{\otimes 2} \simeq \Bbbk[T, v]$, where the operations $(\cdot\ {}_n\ \cdot)$ are defined as follows:

$$T^{(r)}f(v)\ {}_n\ T^{(s)}h(v) = \sum_{t\geq 0} (-1)^r \binom{n}{r}\binom{n-r}{t} T^{(s-t)} f(v)\partial^{n-r-t}h(v),$$

$r, s \in \mathbb{Z}_+$, $f, h \in \Bbbk[v]$. Hereinafter, $T^{(m)} = T^m/m!$ (it is suitable to suppose $T^{(m)} = 0$ for $m < 0$), $\partial = d/dv$ is the ordinary derivation with respect to the variable v.

The collection of operators $W = \{(a\ {}_n\ \cdot) \in \operatorname{End}\mathcal{W} : a \in \mathcal{W}, n \in \mathbb{Z}_+\}$ is a subalgebra of the algebra of linear transformations of the space $\mathcal{W}$, moreover, $W \simeq A_1 = \Bbbk\langle x, d \mid dx - xd = 1\rangle$, i.e., W is isomorphic to the first Weyl algebra (this is a reason for the name of the conformal algebra $\mathcal{W}$).

The canonical representation of A_1 on the space of polynomials gives rise to the natural action of the conformal algebra $\mathcal{W}$ on $\Bbbk[T]$. Namely, the family of operations $(\cdot\ {}_n\ \cdot) : \mathcal{W} \otimes \Bbbk[T] \to \Bbbk[T]$, $n \in \mathbb{Z}_+$, given by

$$T^{(r)}f(v)\ {}_n\ T^{(s)} = (-1)^r \binom{n}{r} T^{(s+r-n)} f(T), \quad f \in \Bbbk[v],\ r, s \geq 0,$$

satisfies the conditions similar to (C1) and (C2).

For an integer $N \geq 1$, the set $\mathbb{M}_n(\mathcal{W})$ of all matrices of size N over $\mathcal{W}$ is also a conformal algebra (denoted by Cend_N [1, 4]) which acts on the space of columns $M_N = \Bbbk[T] \otimes \Bbbk^N$ by the ordinary matrix multiplication rule.

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In [1], V. Kac stated the problem to describe *irreducible* conformal subalgebras $C \subseteq \mathrm{Cend}_N$, i.e., such that there are no non-trivial $\Bbbk[T]$-submodules of M_N invariant with respect to the operators $(a\,{}_n\,\cdot)$, $a \in C$, $n \in \mathbb{Z}_+$. In [5], such a description was obtained for $N = 1$ and also for the case when C is a finitely generated module over $\Bbbk[T]$. In the same paper, a conjecture on the structure of irreducible subalgebras of Cend_n was stated. The conjecture was proved in [6].

The classification of irreducible conformal subalgebras of Cend_N leads to a description of a class of "good" subalgebras of the matrix Weyl algebra $\mathbb{M}_N(A_1)$ acting irreducibly on M_N (see [7]). Note that the problem of description of all such subalgebras of $\mathbb{M}_N(A_1)$ remains open even for $N = 1$.

In the present paper, we consider the notion of a conformal algebra over a linear algebraic group G. The class of such algebras includes ordinary algebras over a field (for $G = \{e\}$) and conformal algebras (for $G = \mathbb{A}^1 \simeq (\Bbbk, +)$). On the other hand, the notion of a conformal algebra over G is equivalent to the notion of a pseudo-algebra [8] over H, where $H = \Bbbk[G]$ is the Hopf algebra of regular functions (coordinate algebra) on G.

The analogue of Cend_N in the class of conformal algebras over G can be considered as a collection of transformation rules (satisfying certain conditions) of the space of vector-valued regular functions $u : G \to \Bbbk^N$ by means of elements of the group G. For example, the left-shift transformation $L : \gamma \mapsto L_\gamma$, where $(L_\gamma u)(x) = u(\gamma x)$, $\gamma, x \in G$, belongs to Cend_N.

We will prove a generalization of the main result of [6] for the case of an arbitrary linear algebraic group G. For $G = \{e\}$ this would be the classical Burnside Theorem, and if $G = \mathbb{A}^1$ then it turns out to be crucial point of proof of the conjecture from [5]. We will apply the result obtained in the case of "intermediate" complexity to describe irreducible conformal subalgebras over a finite group.

1. **Multicategories and operads.** In this section, we state some notions related to operads [9] and multicategories (also known as pseudo-tensor categories [10]).

An ordered n-tuple of integers $\pi = (m_1, \dots, m_n)$, $m_i \geq 1$, is said to be an *n-partition* of m if $m_1 + \dots + m_n = m$. The set of all such partitions is denoted by $\Pi(m, n)$. Each partition $\pi \in \Pi(m, n)$ defines a bijective correspondence between the sets $\{1, \dots, m\}$ and $\{(i, j) \mid i = 1, \dots, n,\ j = 1, \dots, m_i\}$, namely,

$$(i, j) \leftrightarrow (i, j)^\pi := m_1 + \dots + m_{i-1} + j.$$

For two partitions

$$\tau = (p_1, \dots, p_m) \in \Pi(p, m), \quad \pi = (m_1, \dots, m_n) \in \Pi(m, n),$$

define $\tau\pi \in \Pi(p, n)$ in the following way:

$$\tau\pi = (p_1 + \dots + p_{m_1}, p_{m_1+1} + \dots + p_{m_1+m_2}, \dots, p_{m-m_n+1} + \dots + p_m).$$

If $\tau\pi = (q_1, \dots, q_n)$ then for every $i = 1, \dots, n$ let $\tau\pi_i$ denotes the subpartition $(p_{m_1+\dots+m_{i-1}+1}, \dots, p_{m_1+\dots+m_i}) \in \Pi(q_i, m_i)$.

Suppose $\mathcal{A}$ is a class of objects such that for every integer $n \geq 1$ and for every family $A_1, \dots, A_n, A \in \mathcal{A}$ there exists a linear space $P_n^{\mathcal{A}}(A_1, \dots, A_n; A) = P_n(\{A_i\}; A)$ over a fixed field $\Bbbk$.

Also, suppose that for every $A_1, \dots, A_m \in \mathcal{A}$, $B_1, \dots, B_n \in \mathcal{A}$, $C \in \mathcal{A}$ and for every n-partition $\pi = (m_1, \dots, m_n)$ of m there exists a linear map

$$\mathrm{Comp}^\pi : P_n(\{B_i\}; C) \otimes \bigotimes_{i=1}^{n} P_{m_i}(\{A_{(i,j)^\pi}\}; B_i) \to P_m(A_1, \dots A_m; C). \qquad (1)$$

To shorten the notation, we will denote $\mathrm{Comp}^\pi(\varphi, \psi_1, \dots, \psi_n)$ by $\mathrm{Comp}^\pi(\varphi, \{\psi_i\})$ or by $\varphi(\psi_1, \dots, \psi_n)$, if the structure of a partition π is clear.

The elements of the spaces $P_n(\{A_i\}; B)$ are called *multimorphisms* (or *n-morphisms*), the family of maps Comp^π, $\pi \in \Pi(m,n)$, $m,n \geq 1$, is called a *composition rule.*

A class $\mathcal{A}$ endowed with spaces of multimorphisms and a composition rule is called a *multicategory* if the following axioms hold.

(A1) The composition rule is associative. The latter means that, given a collection of objects $A_h, B_j, C_i \in \mathcal{A}$ ($h = 1,\dots,p$, $j = 1,\dots,m$, $i = 1,\dots,n$), two partitions $\tau = (p_1,\dots,p_m) \in \Pi(p,m)$, $\pi = (m_1,\dots,m_n) \in \Pi(m,n)$, an object $D \in \mathcal{A}$, an n-morphism $\varphi \in P_n(\{C_i\}; D)$, and two collections of multimorphisms

$$\psi_j \in P_{p_j}(\{A_{(j,t)^\tau}\}; B_j),\ j = 1,\dots,m, \quad \chi_i \in P_{m_i}(\{B_{(i,t)^\pi}\}; C_i),\ i = 1,\dots,n,$$

we have

$$\mathrm{Comp}^\tau\big(\mathrm{Comp}^\pi(\varphi,\{\chi_i\}),\{\psi_j\}\big) = \mathrm{Comp}^{\pi\tau}\big(\varphi,\big\{\mathrm{Comp}^{\tau_i}(\chi_i,\{\psi_{(i,t)}\})\big\}\big),$$

where $\tau_i = (p_{i1},\dots,p_{im_i})$ are subpartitions of τ.

(A2) For every $A \in \mathcal{A}$ there exists an "identity" 1-morphism $\mathrm{id}_A \in P_1(A;A)$ such that

$$f(\mathrm{id}_{A_1},\dots,\mathrm{id}_{A_n}) = \mathrm{id}_A(f) = f$$

for all $f \in P_n(\{A_i\};A)$, $n \geq 1$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two multicategories. A *functor* from $\mathcal{A}$ to $\mathcal{B}$ is a rule F which maps an object $A \in \mathcal{A}$ to $F(A) \in \mathcal{B}$ in such a way that for every $\varphi \in P_n^{\mathcal{A}}(A_1,\dots,A_n;A)$ there exists $F(\varphi) \in P_n^{\mathcal{B}}(F(A_1),\dots,F(A_n);F(A))$, where

$$F(\mathrm{Comp}^\pi(\varphi,\{\psi_i\})) = \mathrm{Comp}^\pi(F(\varphi),\{F(\psi_i)\});$$
$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)},$$

and the map $\varphi \mapsto F(\varphi)$ is linear.

One of the most natural examples of a multicategory is the class $\mathrm{Vec}_{\Bbbk}$ of linear spaces over a field $\Bbbk$ with respect to

$$P_n^{\mathrm{Vec}_{\Bbbk}}(A_1,\dots,A_n;A) = \mathrm{Hom}(A_1\otimes\dots\otimes A_n, A),$$

where the rule Comp is the ordinary composition of multilinear maps.

The multicategory $\mathrm{Vec}_{\Bbbk}$ can be considered as a particular case of multicategory $\mathcal{M}^*(H)$ of left unital modules over a (coassociative) bialgebra H (see [8] for details).

Let us consider one more example of a multicategory. For every integer $n \geq 1$ denote by $\mathrm{Alg}(n)$ the linear space spanned by all binary trees with n leaves. Such a tree can be naturally identified with a bracketing on the word $x_1\dots x_n$ over the alphabet $X = \{x_1, x_2, \dots\}$. The composition of binary trees can be expressed as (1) by means of the composition of words as follows:

$$\begin{aligned}&\mathrm{Comp}^\pi(u, v_1,\dots,v_n)\\ &\qquad = u(v_1(x_{(1,1)^\pi},\dots,x_{(1,m_1)^\pi}),\dots,v_n(x_{(n,1)^\pi},\dots,x_{(n,m_n)^\pi}))\end{aligned} \tag{2}$$

where $\pi \in \Pi(m,n)$, $u = u(x_1,\dots,x_n) \in \mathrm{Alg}(n)$, $v_i = v_i(x_1,\dots,x_{m_i}) \in \mathrm{Alg}(m_i)$, $i = 1,\dots,n$.

The class Alg which consists of a single object endowed with multimorphisms $P_n = \mathrm{Alg}(n)$ and the composition rule (2) is a multicategory. A multicategory with a single object is known as *operad.*

Definition 1 (see, e.g., [9])**.** An *algebra* in a multicategory $\mathcal{A}$ is a functor F from the operad Alg to $\mathcal{A}$.

Every functor F from Alg to $\mathcal{A}$ is completely defined by an object $A \in \mathcal{A}$ and by a 2-morphism $\mu = F(x_1x_2) \in P_2^{\mathcal{A}}(A,A;A)$. If $\mathcal{A} = \mathrm{Vec}_{\Bbbk}$ then μ is an ordinary product; if $\mathcal{A} = \mathcal{M}^*(H)$ then μ is called a *pseudo-product.* Algebras in the

multicategory $\mathcal{M}^*(H)$ are known as *pseudo-algebras* over H, or H-*pseudo-algebras* [8].

An algebra F in the sense of Definition 1 is said to be *associative* if

$$F(x_1(x_2x_3)) = F((x_1x_2)x_3).$$

In order to define what is, for example, a commutative or Lie algebra, one needs the notion of a symmetric multicategory. In the present paper we consider associative algebras only, so we do not need a symmetric structure.

2. **Conformal algebras over a linear algebraic group.** Consider a linear algebraic group G and the corresponding coordinate Hopf algebra $H = \Bbbk[G]$ with coproduct Δ, counit ε, and antipode S. We will use the short Sweedler notation as follows:

$$\Delta(h) = h_{(1)} \otimes h_{(2)}, \quad (\Delta\otimes \mathrm{id}_H)\Delta(h) = (\mathrm{id}_H \otimes\Delta)\Delta(h) = h_{(1)} \otimes h_{(2)} \otimes h_{(3)},$$
$$(S\otimes \mathrm{id}_H)\Delta(h) = h_{(-1)} \otimes h_{(2)}, \quad (\mathrm{id}_H \otimes S)\Delta(h) = h_{(1)} \otimes h_{(-2)},$$

and so on.

Denote by L_g, $g \in G$, the operator of left shift on H, i.e., $L_gh = h_{(1)}(g)h_{(2)}$, $h \in H$. It is clear that $L_{g_1}L_{g_2} = L_{g_2g_1}$.

Introduce the following multicategory structure on the class of all left unital modules over the algebra H. Let $M_1,\dots,M_n,M$ be such modules, and let $P_n(M_1,\dots,M_n;M)$ stands for the space of all functions

$$a : G^{n-1} \to \mathrm{Hom}_{\Bbbk}(M_1 \otimes \dots \otimes M_n, M)$$

such that

- for every $u_i \in M_i$, $i = 1,\dots,n$, the map $G^{n-1} \to M$ defined by $x \mapsto a(x)(u_1,\dots,u_n)$ is a regular M-valued function on G^{n-1};
- for every $u_i \in M_i$, $f_i \in H$, $i = 1,\dots,n$, $g_1,\dots,g_{n-1} \in G$ we have

$$a(g_1,\dots,g_{n-1})(f_1u_1,\dots,f_nu_n)$$
$$= f_1\big(g_1^{-1}\big)\dots f_{n-1}\big(g_{n-1}^{-1}\big)(L_{g_1}\dots L_{g_{n-1}}f_n)a(g_1,\dots,g_{n-1})(u_1,\dots,u_n).$$

Define the following composition rule on the spaces mentioned above. Suppose we are given $\pi = (m_1,\dots,m_n) \in \Pi(m,n)$, $\psi_i \in P_{m_i}(\{N_{(i,j)^\pi}\};M_i)$, $i = 1,\dots,n$, $\varphi \in P_n(\{M_i\};M)$. Consider a family of $g_j \in G$, $j = 1,\dots,m-1$, and put $\gamma_i = g_{(i,m_i)^\pi}\dots g_{(i,1)^\pi} \in G$ for $i = 1,\dots,n-1$, $\bar g_i = (g_{(i,1)^\pi}\dots g_{(i,m_i-1)^\pi}) \in G^{m_i-1}$ for $i = 1,\dots,n$ (if $m_i = 1$ then $\bar g_i$ is void), $\bar\gamma = (\gamma_1,\dots,\gamma_{n-1}) \in G^{n-1}$. Now, define

$$\mathrm{Comp}^\pi(\varphi,\psi_1,\dots,\psi_n)(g_1,\dots,g_{m-1}) = \mathrm{Comp}^\pi(\varphi(\bar\gamma),\psi_1(\bar g_1),\dots,\psi_n(\bar g_n)), \quad (3)$$

where Comp in the right-hand side means the composition rule of the multicategory $\mathrm{Vec}_{\Bbbk}$.

Let us denote the multicategory constructed above by $\mathcal{M}^*(H)$, since it is not difficult to note that this is a particular case of the multicategory from [8].

Definition 2. A *conformal algebra* over a linear algebraic group G is an algebra in the multicategory $\mathcal{M}^*(H)$, $H = \Bbbk[G]$.

In the language of "ordinary" algebraic operations a conformal algebra over G can be defined as a left H-module C endowed with a family of $\Bbbk$-linear operations $(\cdot\,{}_\gamma\,\cdot) : C \otimes C \to C$, $\gamma \in G$, such that for every $a, b \in C$, $h \in H$, $g \in G$ we have:

(G1) the map $(a\,{}_x\,b) : \gamma \mapsto (a\,{}_\gamma\,b)$ is a regular C-valued function on G;
(G2) $(ha\,{}_g\,b) = h(g^{-1})(a\,{}_g\,b)$;
(G3) $(a\,{}_g\,hb) = L_gh(a\,{}_g\,b)$.

A conformal algebra over a trivial group $G = \{e\}$ is just an ordinary algebra over the field $\Bbbk$; if $G = \mathbb{A}^1$ (the affine line) then we get the definition of a conformal algebra [1] in terms of λ-brackets; in the case $G = \mathrm{GL}_1(\Bbbk)$ we obtain the notion of a $\mathbb{Z}$-conformal algebra [11].

The associativity property of a conformal algebra can be easily expressed in terms of the operations $(\cdot \, {}_g \, \cdot)$ by means of computation the images of $x_1(x_2x_3) = \mathrm{Comp}^{(1,2)}(x_1x_2, x_1, x_1x_2)$ and $(x_1x_2)x_3 = \mathrm{Comp}^{(2,1)}(x_1x_2, x_1x_2, x_1)$. According to (3), a conformal algebra C over G is associative if and only if

$$a \, {}_g \, (b \, {}_\gamma \, c) = (a \, {}_g \, b) \, {}_{\gamma g} \, c \tag{4}$$

for all $a, b, c \in C$, $g, \gamma \in G$.

Example 1. Let A be an H-comodule algebra, i.e., a (not necessarily associative) algebra equipped by a coaction map $\Delta_A : A \to H \otimes A$ such that

- Δ_A is a homomorphism of algebras;
- $(\Delta \otimes \mathrm{id}_A)\Delta_A = (\mathrm{id}_H \otimes \Delta_A)\Delta_A$.

We will use the Sweedler notation for Δ_A as well as for Δ. Then the free H-module $C = H \otimes A$ under the operations

$$(h \otimes a) \, {}_\gamma \, (f \otimes b) = h(\gamma^{-1}) b_{(1)}(\gamma) L_\gamma f \otimes ab_{(2)}, \quad \gamma \in G, \ f, h \in H, \ a, b \in A, \tag{5}$$

is a conformal algebra over G. Moreover, if A is associative then C satisfies (4). Following [12], denote the conformal algebra C obtained by $\mathrm{Diff}(A, \Delta_A)$.

Note that an arbitrary algebra A is an H-comodule algebra with respect to the coaction $\Delta_A^0(a) = 1 \otimes a$, $a \in A$. The conformal algebra $\mathrm{Diff}(A, \Delta_A^0)$ is called the loop algebra or current algebra over A, denoted by $\mathrm{Cur}^H A$.

3. **Conformal linear maps.** Let M and N be linear spaces over the field $\Bbbk$. The set $\mathrm{Hom}(M, N)$ is a topological linear space with respect to the finite topology (see, e.g., [13]). In that sense, a sequence $\{\alpha_k\}_{k \geq 0} \subset \mathrm{Hom}(M, N)$ converges to a map $\alpha \in \mathrm{Hom}(M, N)$ if and only if for every finite number of elements $u_1, \dots, u_n \in M$ there exists a natural m such that $\alpha_k(u_i) = \alpha(u_i)$, $i = 1, \dots, n$, for all $k \geq m$.

Definition 3. A map $a : G \to \mathrm{Hom}(M, N)$ is said to be *locally regular* if for every $u \in M$ the map $z \mapsto a(z)u$, $z \in G$, is a regular N-valued function on G.

Let $\{h_i\}_{i \in I}$ be a linear basis of H over $\Bbbk$. Consider the dual space H^* endowed with a topology defined by the basic neighborhoods of zero of the form

$$X(i_1, \dots, i_n) = (\mathrm{Span}_\Bbbk\{h_{i_1}, \dots, h_{i_n}\})^\perp \subseteq H^*, \quad i_k \in I, \ k = 1, \dots, n, \ n \geq 0.$$

It is clear that, up to equivalence, the topology does not depend on the choice of basis in H.

Suppose $a : G \to \mathrm{Hom}(M, N)$ is a locally regular map. Then for every $u \in M$ the function $a(x)u : G \to N$ can be presented as $a(x)u = \sum_{i \in I} h_i \otimes v_i \in H \otimes N$, where $a(z)u = \sum_{i \in I} h_i(z) v_i$. For any $\xi \in H^*$ define $\tilde{a}(\xi) \in \mathrm{Hom}(M, N)$ in the following way:

$$\tilde{a}(\xi) : u \mapsto (\langle \xi, \cdot \rangle \otimes \mathrm{id})(a(x)u), \quad u \in M.$$

Let us denote by $\tilde{a}$ the linear map $H^* \to \mathrm{Hom}(M, N)$ which maps ξ to $\tilde{a}(\xi)$.

Lemma 1. 1. *If a is locally regular then $\tilde{a}$ is continuous with respect to the finite topology on* $\mathrm{Hom}(M, N)$.

2. *For any continuous linear map $\alpha : H^* \to \mathrm{Hom}(M, N)$ there exists a locally regular $a : G \to \mathrm{Hom}(M, N)$ such that $\tilde{a} = \alpha$.*

Proof. 1. Due to the local regularity of a, for every $u_1,\dots,u_n\in M$ there exist only a finite number of basic $h_{i_1},\dots,h_{i_m}\in H$ that appear in the presentations of $a(x)u_j$, $j=1,\dots,n$. Therefore, if $\xi\in X(i_1,\dots,i_m)$ then $\tilde a(\xi)u_j=0$ for all $j=1,\dots,n$, i.e., $\tilde a(\xi)\to 0$ as $\xi\to 0$ with respect to the finite topology on $\mathrm{Hom}(M,N)$.

2. For each $i\in I$ denote by ξ_i the element of H^* defied by the rule $\langle \xi_i,h_j\rangle=\delta_{i,j}$, $i,j\in I$. For a fixed $u\in M$ the set $\{i\in I\mid \alpha(\xi_i)u\neq 0\}$ is finite. Hence, we may consider the map $a(g)\in\mathrm{Hom}(M,N)$, $g\in G$, given as follows:

$$a(g)u=\sum_{i\in I}h_i(g)\alpha(\xi_i)u.$$

The function $a(x)u: g\mapsto a(g)u$ is regular, and for the locally regular $a: G\to \mathrm{Hom}(M,N)$ we have $\tilde a=\alpha$. □

Let G_1, G_2 be two linear algebraic groups, $H_i=\Bbbk[G_i]$, $i=1,2$, be the corresponding Hopf algebras, then $\Bbbk[G_1\times G_2]=H_1\otimes H_2$.

Suppose we are given a locally regular map $a: G_1\times G_2\to\mathrm{Hom}(M,N)$. For every $\xi\in H_1^*$, $u\in M$ define

$$\tilde a(\xi,y)u=(\langle\xi,\cdot\rangle\otimes\mathrm{id}_{H_2}\otimes\mathrm{id}_N)(a(x,y)u),$$

where (x,y) is a variable that ranges over $G_1\times G_2$. We obtain a function

$$\tilde a: H_1^*\times G_2\to\mathrm{Hom}(M,N),\quad (\xi,z)\mapsto\tilde a(\xi,z),\quad \xi\in H_1^*,\ z\in G_2.$$

Lemma 2. *For every $z\in G_2$ the map $\xi\mapsto\tilde a(\xi,z)$ is continuous. For every $\xi\in H_1^*$ the map $z\mapsto\tilde a(\xi,z)$ is locally regular.*

Proof. If $u\in M$, $\{h_i\}$ and $\{f_j\}$ are bases of H_1 and H_2, respectively, then $a(x,y)u=\sum\limits_{i,j}h_i(x)\otimes f_j(y)\otimes v_{ij}$, where the sum is finite. Both statements obviously follow from this presentation. □

Let V be a G-set, i.e., a Zariski closed subset of an affine space endowed with a continuous action of the group G. The algebra A of regular functions on V is an H-comodule algebra with the coaction $\Delta_A: A\to H\otimes A$ dual to the action of G on V. There exists a natural (right) representation of G on A by the left-shift automorphisms, namely, $L: g\mapsto L_g$, $g\in G$, where $(L_gf)(v)=f(gv)$, $f\in A$, $v\in V$.

The representation L, in particular, has the following properties:

- for any $f\in A$ the map $g\mapsto L_gf$ is an A-valued regular map on G;
- $L_g(f_1f_2)=L_g(f_1)L_g(f_2)$ for all $f_1,f_2\in A$, $g\in G$.

Consider the space of operators that have the same properties as L.

Definition 4. A *conformal linear transformation* (*conformal endomorphism*) of an A-module M is a locally regular map $a: G\to\mathrm{End}\,M$ such that

$$a(g)(fu)=L_gf(a(g)u),\quad u\in M,\ f\in A,\ g\in G. \tag{6}$$

The property (6) is called *translation invariance* or T-invariance of the map a.

Denote by $\mathrm{Cend}^{G,V}M$ the space of all conformal endomorphisms of an A-module M.

Introduce the following structure of an H-module on $C=\mathrm{Cend}^{G,V}M$:

$$(fa)(g)=f(g^{-1})a(g),\quad f\in H,\ a\in C,\ g\in G. \tag{7}$$

Also, consider the operations $(\cdot\ {}_g\ \cdot)$, $g\in G$, given by

$$(a\ {}_g\ b)(z)=a(g)b(zg^{-1}),\quad a,b\in C,\ g,z\in G \tag{8}$$

(it is easy to check that $(a\ {}_g\ b)$ is a locally regular map satisfying (6)).

Proposition 1. *The space* $\mathrm{Cend}^{G,V} M$*, where* M *is a finitely generated* A*-module, is an associative conformal algebra over* G *with respect to the operations* (7)*,* (8)*.*

Proof. Let us check the conditions (G1)–(G3) for the H-module $C = \mathrm{Cend}^{G,V} M$. It follows from (6) that (7), (8) satisfy (G2) and (G3).

To check (G1), consider $u \in M$, $a, b \in C$, $z, g \in G$. Relation (8) implies that the map $c : (g, z) \mapsto (a\ {}_g\ b)(z) \in \mathrm{End}\, M$ is a locally regular function on $G \times G$. Hence, by Lemma 2, for every $\xi \in H^*$ the map $\tilde{c}(\xi, x) : G \to \mathrm{End}\, M$, $z \mapsto \tilde{c}(\xi, z)$, locally regular. It is also clear that $\tilde{c}(\xi, x)$ is T-invariant. Therefore, $\tilde{c}(\xi, x) \in C$.

Suppose $e_1, \dots, e_n \in M$ is a set of generators of the H-module M. There exist finite collections of elements $f^1_{ik}, h^1_{ik}, f^2_{ik}, h^2_{ik} \in H$, $i, k = 1, \dots, n$, such that

$$b(x)e_i = \sum_{k=1}^{n} f^1_{ik}(x) \otimes f^2_{ik} e_k, \quad a(x)e_i = \sum_{k=1}^{n} h^1_{ik}(x) \otimes h^2_{ik} e_k.$$

Then

$$c(x, y)e_i = (a\ {}_x\ b)(y)e_i = a(x)b(yx^{-1})e_i = \sum_{k,l=1}^{n} f^1_{ik}(yx^{-1}) h^1_{kl}(x)(L_x f^2_{ik}) h^2_{kl} e_l$$
$$= \sum_{k,l=1}^{n} f^1_{ik(-2)} f^2_{ik(1)} h_{kl} \otimes f^1_{ik(1)} \otimes f^2_{ik(2)} h^2_{kl} e_l.$$

If $\xi \in H^*$ is sufficiently close to zero then $\tilde{c}(\xi, z)e_i = 0$ for all $z \in G$ and for all $i = 1, \dots, n$. Therefore, $\tilde{c}(\xi, x) = 0$, and the function $\tilde{c} : H^* \to C$, $\xi \mapsto \tilde{c}(\xi, x) \in C$, is continuous with respect to the discrete topology on C.

Let us fix $a \in C$, $\xi \in H^*$ and denote by $\tilde{a}(\xi) \in \mathrm{End}\, C$ the linear map that turns $b \in C$ into $\tilde{c}(\xi, x) \in C$, where $\tilde{c}$ is constructed from $a, b \in C$ as above. Since $\tilde{c}$ is continuous with respect to the discrete topology on C, the map $\alpha : \xi \mapsto a(\xi)$ is continuous with respect to the finite topology on $\mathrm{End}\, C$. By Lemma 1(2), there exists a locally regular map $a_1 : G \to \mathrm{End}\, C$ such that $\tilde{a}_1(\xi) = \alpha(\xi)$ for every $\xi \in H^*$.

Note that for all $g, z \in G$, $b \in C$, $u \in M$ we have

$$(a_1(g)b)(z)u = \sum_{i \in I} h_i(g)(\alpha(\xi)b)(z)u = \sum_{i \in I} h_i(g)(\langle \xi_i, \cdot \rangle \otimes \mathrm{ev}_z \otimes \mathrm{id})c(x, y)u$$
$$= (\mathrm{ev}_g \otimes \mathrm{ev}_z \otimes \mathrm{id})c(x, y)u = (a\ {}_g\ b)(z)u,$$

since $\sum\limits_{i \in I} h_i(g)\langle \xi_i, \cdot \rangle = \mathrm{ev}_g$. Therefore, $a_1(g)b = (a\ {}_g\ b)$ and the function $(a\ {}_x\ b)$ is regular.

Associativity of the conformal algebra $\mathrm{Cend}^{G,V} M$ follows immediately from the definition of operations (8). □

The most interesting case is when M is a free A-module. Then one may identify M with the space of regular vector-valued functions on V, and $\mathrm{Cend}^{G,V} M$ is a collection of transformation rules of this space by means of the group G (e.g., the left shift L belongs to $\mathrm{Cend}^{G,V} M$).

If M is a free n-generated A-module then let us denote $\mathrm{Cend}^{G,V} M$ by $\mathrm{Cend}^{G,V}_n$. The structure of the conformal algebra $\mathrm{Cend}^{G,V}_n$ is completely described by the following statement.

Theorem 1. *The conformal algebra* $\mathrm{Cend}^{G,V}_n$ *is isomorphic to the conformal algebra* $\mathrm{Diff}(A \otimes \mathbb{M}_n(\Bbbk), \Delta_A \otimes \mathrm{id})$*.*

Proof. Let us fix a basis $\{e_k\}_{k=1}^n$ of a free A-module M. Then M is isomorphic to $A \otimes \Bbbk^n$.

An arbitrary element $a \in \mathrm{Cend}_n^{G,V}$ is uniquely defined by a collection of regular M-valued functions $a(x)e_k \in H \otimes M$, $k = 1, \dots, n$. Assume

$$a(x)e_k = \sum_{i\in I} h_i \otimes u_{ik}, \quad u_{ik} = \sum_{j\in J} f_j \otimes v_{ijk},$$

where $\{h_i\}_{i\in I}$ is a basis of H, $\{f_j\}_{j\in J}$ is a basis of A, $v_{ijk} \in \Bbbk^n$. Then

$$a(z)\left(\sum_{k=1}^n g_k e_k\right) = \sum_{i\in I, j\in J} \sum_{k=1}^n h_i(z) f_j L_z g_k v_{ijk}, \quad g_k \in A.$$

Consider the linear maps $a_{ij} \in \operatorname{End} \Bbbk^n \simeq \mathbb{M}_n(\Bbbk)$ that are defined by their values on the canonical basis: $a_{ij}e_k = v_{ijk}$, $k = 1, \dots, n$. Denote

$$C = \operatorname{Diff}(A \otimes \mathbb{M}_n(\Bbbk), \Delta_A \otimes \mathrm{id}) = H \otimes A \otimes \mathbb{M}_n(\Bbbk),$$

and define

$$\Phi : a \mapsto \Phi(a) = \sum_{i\in I, j\in J} S(h_i) \otimes f_j \otimes a_{ij} \in C.$$

The map $\Phi : \mathrm{Cend}_n^{G,V} \to C$ constructed is H-linear, and for all $a, b \in \mathrm{Cend}_n^{G,V}$, $z \in G$ we have

$$(a \;{}_z\; b)(x)e_k = a(z)b(xz^{-1})e_k = \sum_{i\in I, j\in J} h_{i(1)}h_{i(-2)}(z) \otimes a(z)(f_j b_{ij} e_k)$$

$$= \sum_{i,l\in I,\, j,p\in J} h_l(z)h_{i(1)}h_{i(-2)}(z) \otimes f_p L_z f_j \otimes a_{lp} b_{ij} e_k = (\Phi(a) \;{}_z\; \Phi(b))(x)e_k,$$

where the right-hand side is computed by (5). Therefore, $\Phi(a \;{}_z\; b) = \Phi(a) \;{}_z\; \Phi(b)$, $z \in G$, so Φ is a homomorphism of conformal algebras. The inverse map Φ^{-1} is given by the rule

$$\sum_{i\in I, j\in J} h_i \otimes f_j \otimes a_{ij} \mapsto a \in \mathrm{Cend}_n^{G,V},$$

where $a(z) = \sum\limits_{i\in I, j\in J} h_i(z^{-1}) f_j L_z \otimes a_{ij} \in \operatorname{End} M$. Hence, Φ is an isomorphism. $\square$

Hereinafter, we identify $\mathrm{Cend}_n^{G,V}$ and the conformal algebra $H \otimes A \otimes \mathbb{M}_n(\Bbbk) \simeq H \otimes \mathbb{M}_n(A)$ with operations (5).

Define an H-linear map $\mathcal{F} : H \otimes A \to H \otimes A$ as follows: $\mathcal{F}(f \otimes a) = f a_{(-1)} \otimes a_{(2)}$. This map is invertible: $\mathcal{F}^{-1}(h \otimes a) = h a_{(1)} \otimes a_{(2)}$. We will also denote by $\mathcal{F}$ the linear transformation $\mathcal{F} \otimes \mathrm{id}$ of the space $H \otimes A \otimes \mathbb{M}_n(\Bbbk) \simeq H \otimes \mathbb{M}_n(A)$.

Proposition 2. 1. *A right ideal of* $\mathrm{Cend}_n^{G,V}$ *is of the form* $B = H \otimes B_0$*, where* B_0 *is a right ideal of* $\mathbb{M}_n(A)$.
2. *A left ideal of* $\mathrm{Cend}_n^{G,V}$ *is of the form* $B = \mathcal{F}(H \otimes B_0)$*, where* B_0 *is a left ideal of* $\mathbb{M}_n(A)$,
3. *Conformal algebra* $\mathrm{Cend}_n^{G,V}$ *is simple if and only if* G *acts transitively on* V.

Proof. Statements 1 and 2 can be proved in a routine way in accordance with a scheme from [5].

To prove the statement 3, assume B is a two-sided ideal of $\mathrm{Cend}_n^{G,V}$. Then $B = H \otimes B_0$ as a right ideal, and B_0 has to be a two-sided ideal invariant with respect to the left-shift action of G. This implies $B_0 = A_0 \otimes \mathbb{M}_n(\Bbbk)$, where A_0 is a G-invariant ideal of $A = \Bbbk[V]$. The converse is also true: if A_0 is a G-invariant ideal of A then $B = H \otimes A_0 \otimes \mathbb{M}_n(\Bbbk)$ is a (two-sided) ideal of $\mathrm{Cend}_n^{G,V}$. It remains to note that the coordinate algebra of a G-set V contains no non-trivial G-invariant ideals if and only if G acts transitively on V. $\square$

4. **Algebra of operators.** Consider the case when $V = G$ and G acts on V by left multiplications. Then $A = H$, $\Delta_A = \Delta$. Let us denote by Cend_n the conformal algebra $\mathrm{Cend}_n^{G,G}$. The main investigation tool of the conformal algebra Cend_n is the (ordinary) algebra generated by linear operators of the form $a(g) \in \mathrm{End}\, M_n$, $a \in \mathrm{Cend}_n$, $g \in G$. Throughout the rest of the paper, the field $\Bbbk$ is supposed to be algebraically closed.

As above, we identify a free n-generated H-module M_n with $H \otimes \Bbbk^n$ choosing a basis.

Denote by W_n the linear span in $\mathrm{End}\, M_n$ of all operators of the form $a(g)$, $a \in \mathrm{Cend}_n$, $g \in G$. This is an (ordinary) associative subalgebra of $\mathrm{End}\, M_n$ since (8) implies

$$a(g)b(z) = (a \,{}_g\, b)(zg), \quad g_1, g_2 \in G. \tag{9}$$

Algebra $W_n \subseteq \mathrm{End}\, M_n$ is equipped by the induced finite topology.

Example 2. It is easy to see that if $a = \sum_{i,j\in I} h_i \otimes h_j \otimes a_{ij} \in \mathrm{Cend}_n$ then $a(g) = \sum_{i,j\in I} h_i(g^{-1}) h_j L_g \otimes a_{ij}$. Therefore,

1) if $G = \{e\}$ then $W_n = \mathrm{End}\, \Bbbk^n \simeq \mathbb{M}_n(\Bbbk)$;

2) if $G = \mathbb{A}^1$, $\mathrm{char}\, \Bbbk = 0$, then the algebra W_n consists of operators

$$\sum_{i=1}^{m} A_i(x) e^{\lambda_i \partial}, \quad A_i \in \mathbb{M}_n(\Bbbk[x]), \quad \lambda_i \in \Bbbk,\ m \geq 0,$$

that act on $\Bbbk[x] \otimes \Bbbk^n$, where ∂ is the ordinary derivation with respect to x.

Lemma 3. *Suppose $g_1, \dots, g_m$ be pairwise different elements of G, $m \geq 1$, and let $U \subseteq G$ be a nonempty Zariski open subset. Then there exist $f_1, \dots, f_m \in H$ such that*

$$\begin{vmatrix} L_{g_1} f_1 & L_{g_1} f_2 & \dots & L_{g_1} f_m \\ L_{g_2} f_1 & L_{g_2} f_2 & \dots & L_{g_2} f_m \\ \dots & \dots & \dots & \dots \\ L_{g_m} f_1 & L_{g_m} f_2 & \dots & L_{g_m} f_m \end{vmatrix} (z) \neq 0 \tag{10}$$

for some $z \in U$.

Proof. For $m = 1$ the statement is obvious. Assume the lemma is true for a collection of $m-1$ pairwise different $g_2, \dots, g_m \in G$. Denote by h the determinant $|L_{g_i} f_j|_{i,j=2,\dots,m} \in H$. Then $U' = \{z \in U \mid h(z) \neq 0\}$ is a nonempty open subset of G.

Consider an element $g_1 \in G$ different from $g_2, \dots, g_m$; then for any $z \in U'$ the elements $z_i = g_i^{-1} z$, $i = 1, \dots, m$, are pairwise different. Choose $f_1' \in \{f \in H \mid f(z_2) = \dots = f(z_m) = 0,\ f(z_1) \neq 0\}$. Such a function f_1' exists since z_1 does not belong to the closure of $\{z_2, \dots, z_m\} \subset G$. Then $f_1 = L_{g_1^{-1}} f_1'$ satisfies (10). $\square$

The following statement leads to a "normal form" of elements of the algebra W_n.

Lemma 4. 1. *For every $a \in \mathrm{Cend}_n$, $g \in G$ the equality $(a \,{}_g\, \mathrm{Cend}_n) = 0$ holds if and only if $a(g) = 0$.*

2. *If $\sum_i a_i(g_i) = 0$, where $g_i \in G$ are pairwise different, then $a_i(g_i) = 0$ for all i.*

Proof. Statement 1 follows immediately from (9).

To prove 2, suppose $\sum_{i=1}^{m} a_i(g_i) = 0$. Then for all $f \in H$

$$0 = \sum_{i=1}^{m} a_i(g_i) f = \sum_i L_{g_i} f a(g_i)$$

by (6). Assume there exists $u \in M_n$ such that $a(g_k)u \neq 0$ for some k. Fix such an index k and denote by U the set of all $z \in G$ such that $(a(g_k)u)(z) \neq 0$ (we consider M_n as the module of regular $\Bbbk^n$-valued functions on G). The set U is Zariski open, so by Lemma 3 there exist $f_1, \dots, f_m \in H$ such that $h(z) \neq 0$ for some $z \in U$, where $h = \det |L_{g_i} f_j|$. But

$$0 = \sum_{i=1}^{m} (a_i(g_i) f_j u)(z) = \sum_{i=1}^{m} (L_{g_i} f_j)(z)(a(g_i)u)(z), \quad j = 1, \dots, m,$$

so $h(z)(a(g_i)u)(z) = 0$ for all i, therefore, $(a(g_k)u)(z) = 0$. The contradiction obtained proves 2. $\square$

Lemma 5. 1. *Algebra W_n is a topological left H-module with respect to the discrete topology on H and finite topology on W_n.*

2. *Conformal algebra Cend_n is a topological left W_n-module with respect to the finite topology on W_n and discrete topology on Cend_n.*

Proof. It follows from Lemma 4 that the actions of H on W_n and of W_n on Cend_n are well defined by the rules

$$h \cdot a(g) = h(g^{-1}) a(g), \quad a \in \mathrm{Cend}_n,\ h \in H,\ g \in G, \tag{11}$$

$$a(z) \cdot b = (a\ {}_z\ b), \quad a, b \in \mathrm{Cend}_n,\ z \in G. \tag{12}$$

The associativity condition is obvious for (11) and could be easily checked for (12). It remains to show that these maps are continuous.

1. Suppose $W_n \ni \alpha_k \to 0$ as $k \to \infty$. Each α_k can be presented in the normal form from Lemma 4(2). It is necessary to check that $(h \cdot \alpha_k) \to 0$ for a fixed $h \in H$.

Indeed, for every $a \in \mathrm{Cend}_n$, $h \in H$, $g \in G$ we have

$$\begin{aligned} h_{(1)} a(g) h_{(-2)} = h_{(1)} L_g h_{(-2)} a(g) = h_{(1)} h_{(-3)}(g) h_{(-2)} a(g) \\ = \varepsilon(h_{(1)}) h_{(-2)}(g) a(g) = h(g^{-1}) a(g), \end{aligned}$$

therefore,

$$h_{(1)} \alpha_k h_{(-2)} = h \cdot \alpha_k.$$

Since $\Delta(h) = h_{(1)} \otimes h_{(2)}$ involves only a finite number of summands, the sequence $(h \cdot \alpha_k)$ converges to zero in the sense of finite topology on W_n.

2. Consider a sequence $\alpha_k \to 0$ and an element $x \in \mathrm{Cend}_n$. We need to show that $\alpha_k \cdot x = 0$ in Cend_n for sufficiently large k. If $x = 1 \otimes f \otimes a$, $f \in H$, $a \in \mathbb{M}_n(\Bbbk)$, then the claim obviously follow from the definition of finite topology (the action of α_k on x can be considered on the columns of $f \otimes a$ separately). Note that

$$\alpha_k \cdot hx = h_{(2)}((h_{(-1)} \cdot \alpha_k) \cdot x), \quad h \in H,$$

therefore, statement 1 implies $\alpha_k \cdot hx = 0$ as $k \gg 0$. $\square$

Corollary 1. 1. *If $\{\alpha_k\}_{k \geq 0}$ is a fundamental sequence if W_n then for every $a \in \mathrm{Cend}_n$ there exists a natural m such that $\alpha_p \cdot a = \alpha_q \cdot a$ for all $p, q \geq m$.*

2. *If a sequence $\{\alpha_k\}_{k \geq 0}$ converges to $\alpha \in W_n$ as $k \to \infty$ then $\alpha_k \cdot a = \alpha \cdot a$ for any $a \in \mathrm{Cend}_n$ as $k \gg 0$.* $\square$

Proposition 3. *Algebra $W_n \subseteq \operatorname{End} M_n$ acts on M_n irreducibly.*

Proof. Note that for every $f \in H \simeq M_1$, $f \neq 0$, the submodule $W_1 f$ is an ideal of H invariant with respect to left shifts. Hence, $W_1 f = H$.

Consider an arbitrary element $u = \sum\limits_{k=1}^{n} f_k \otimes e_k \in M_n$, $u \neq 0$. Without loss of generality, assume $f_1 \neq 0$. It is easy to see that for every $i = 1, \dots, n$ and for every $h \in H$ there exists $\alpha_{i,h} \in W_n$ such that $\alpha_{i,h} u = h \otimes e_i$: $\alpha_{i,h} = \sum\limits_{j} h_j L_{z_j} \otimes e_{i1}$,

where $\sum\limits_j h_j L_{z_j} f_1 = h$. Therefore, for any $v = \sum\limits_{k=1}^n h_k \otimes e_k \in M_n$ one may build $\alpha = \sum\limits_{k=1}^n \alpha_{k,h_k} \in W_n$ such that $\alpha u = v$. □

Note that W_n contains the operators $\Gamma(h) : M_n \to M_n$, $\Gamma(h)u = hu$, $h \in H$, $u \in M_n$. Indeed, $\Gamma(h) = hL_e = (1 \otimes h \otimes E)(e)$, where E is the identity matrix.

Proposition 4. *Let S be a subalgebra of W_n that acts irreducibly on M_n, and suppose W_n is closed under multiplication by operators $\Gamma(h)$, $h \in H$, on the left and on the right, i.e., $HS, SH \subseteq S$. Then the space of S-invariant linear transformations of the space M_n consists of H-module endomorphisms.*

Proof. Denote

$$D = \operatorname{End}_S M_n := \{\varphi \in \operatorname{End} M_n \mid \varphi\alpha = \alpha\varphi \text{ for all } \alpha \in S\}.$$

By the Schur Lemma, D is a division algebra. Consider arbitrary elements $0 \neq \alpha \in S$, $f \in H$, $\varphi \in D$. Since $f\alpha, \alpha f \in S$, we have

$$(\varphi f)\alpha = \varphi(f\alpha) = (f\alpha)\varphi = (f\varphi)\alpha.$$

Hence,

$$[\varphi, f]\alpha = (\varphi f - f\varphi)\alpha = 0. \tag{13}$$

Note that $\psi = \varphi f - f\varphi \in D$. Indeed,

$$\beta(\varphi f - f\varphi) = (\beta\varphi)f - (\beta f)\varphi = (\varphi\beta)f - \varphi(\beta f) = 0$$

for every $\beta \in S$. Relation (13) implies $[\varphi, f] = 0$, and thus $\varphi \in \operatorname{End}_H M_n$. □

Lemma 6. *If D is a division algebra in $\operatorname{End}_H M_n$ that contains id_{M_n} then $D = \Bbbk$.*

Proof. The endomorphism algebra $\operatorname{End}_H M_n$ of the free H-module M_n is isomorphic to the matrix algebra $\mathbb{M}_n(H)$. Suppose $D \subseteq \mathbb{M}_n(H)$ is a division algebra, and let x stands for a nonzero element of D such that $x \notin \Bbbk E$, where E is the identity matrix.

Since D contains λE for every $\lambda \in \Bbbk$, the element $x + \lambda E \in D$ is nonzero, therefore, invertible in D for all $\lambda \in \Bbbk$. Denote by $f(\cdot, \lambda)$ the function $z \mapsto \det(x + \lambda E)(z)$, $z \in G$. This is a polynomial in λ with coefficients in H. It is clear that $f(\cdot, \lambda) = \lambda^n + \dots + \det x$, $(\det x)(z) \neq 0$ at any point $z \in G$. Since $\Bbbk$ is algebraically closed, for every $z \in G$ there exists $\lambda \in \Bbbk$ such that $f(z, \lambda) = 0$, which is impossible since $x + \lambda E$ is invertible. □

Theorem 2. *If $S \subseteq W_n$ is a subalgebra that acts irreducibly on M_n and $HS, SH \subset S$ then S is a dense (over $\Bbbk$) subalgebra of $\operatorname{End} M_n$.*

Proof. Recall that by the Jacobson Density Theorem (see, e.g., [13]) an algebra A with a faithful irreducible module M and with a centralizer $D = \operatorname{End}_A M$ is a dense subalgebra of $\operatorname{End}_D M$ with respect to the finite topology. The latter means that for an arbitrary linearly independent over D finite set $u_1, \dots, u_m \in M$ and for every $v_1, \dots, v_m \in M$, $m \geq 1$, there exists $\alpha \in A$ such that $\alpha u_i = v_i$, $i = 1, \dots, m$.

By Proposition 4, the centralizer $D = \operatorname{End}_S M_n$ is embedded into $\operatorname{End}_H M_n$. It follows from Lemma 6 that $D = \Bbbk$. Hence, S is dense in $\operatorname{End} M_n$. □

Corollary 2. *The algebra W_n is dense in $\operatorname{End} M_n$.* □

5. **Automorphisms and irreducible subalgebras of the conformal algebra** Cend_n.

Theorem 3. *The group of automorphisms of the conformal algebra* Cend_n *is isomorphic to the group of topological (with respect to the finite topology) H-invariant automorphisms of the algebra W_n.*

Proof. Let Θ be an automorphism of the conformal algebra Cend_n, i.e., a bijective H-invariant map that preserves all operations $(\cdot\ {}_g\ \cdot)$, $g \in G$. Define $\theta : W_n \to W_n$ by the rule

$$\theta(a(g)) = \Theta(a)(g), \quad g \in G, a \in \mathrm{Cend}_n\,. \tag{14}$$

If $a(g) = 0$ then $a\ {}_g\ \mathrm{Cend}_n = 0$, therefore, $\Theta(a)\ {}_g\ \mathrm{Cend}_n = 0$ and $\theta(a(g)) = \Theta(a)(g) = 0$ by Lemma 4(1). It follows from Lemma 4(2) that θ is well-defined on W_n by linearity.

This is straightforward to check that θ is an H-invariant map:

$$\theta(h \cdot a(g)) = \theta(h(g^{-1})a(g)) = h(g^{-1})\Theta(a)(g) = h \cdot (\Theta(a)(g)) = h \cdot \theta(a(g)).$$

To prove the continuity of θ, it is enough to show that for an arbitrary converging sequence $\alpha_k \to \alpha \in W_n$ its image $\{\theta(\alpha_k)\}_{k\ge 0}$ converges to $\theta(\alpha)$ in W_n. Indeed, the definition of θ implies

$$\theta(\alpha_k) \cdot \Theta(a) = \Theta(\alpha_k \cdot a), \quad \theta(\alpha) \cdot \Theta(a) = \Theta(\alpha \cdot a), \quad a \in \mathrm{Cend}_n,\ k \ge 0.$$

Since Θ is surjective, $\theta(\alpha_k)\cdot a = \theta(\alpha)\cdot a$ for every $a \in \mathrm{Cend}_n$, $k \gg 0$, by Lemma 5(2). For the elements of the form $a = (1\otimes f_j \otimes a_j)$, $f_j \in H$, $a_j \in \mathbb{M}_n(\Bbbk)$, one may compute

$$(\theta(\alpha_k) \cdot a)(z)(1 \otimes v) = \theta(\alpha_k)(f_j \otimes a_j v) = 0, \quad v \in \Bbbk^n,\ k \gg 0,$$

therefore, the sequence $\theta(\alpha_k)$ converges to $\theta(\alpha)$.

The map θ has an inverse one that can be easily constructed from Θ^{-1}. Hence, θ is bijective, and its inverse is also continuous.

It is easy to check that all products are preserved:

$$\begin{aligned}\theta(a(g)b(z)) = \theta((a\ {}_g\ b)(zg)) = \Theta(a\ {}_g\ b)(zg) = (\Theta(a)\ {}_g\ \Theta(b))(zg)\\ = \Theta(a)(g)\Theta(b)(z) = \theta(a(g))\theta(b(z)).\end{aligned} \tag{15}$$

Therefore, an arbitrary automorphism Θ of the conformal algebra Cend_n gives rise to a continuous H-invariant automorphism θ of the algebra W_n. It follows from (14) that the map $\Theta \mapsto \theta$ is a group homomorphism.

Conversely, let θ be a continuous H-invariant automorphism of W_n. Consider the map $\Theta : \mathrm{Cend}_n \to \mathrm{Cend}_n$ built by the following rule: for an arbitrary $a \in \mathrm{Cend}_n$ set $\Theta(a) : g \mapsto \theta(a(g))$, $g \in G$.

To complete the proof, it is enough to show that $\Theta(a) \in \mathrm{Cend}_n$ and Θ is an automorphism of Cend_n. For every $h \in H$, $g \in G$ we have

$$\begin{aligned}\Theta(a)(g)h = \theta(a(g))h = h_{(2)}(h_{(-1)} \cdot \theta(a(g))) = h_{(2)}\theta(h_{(-1)} \cdot a(g))\\ = h_{(2)}h_{(1)}(g)\theta(a(g)) = L_g h\Theta(a)(g).\end{aligned}$$

Hence, $\Theta(a)$ is T-invariant. By Corollary 2, the algebra W_n is dense in $\mathrm{End}\, M_n$. Therefore, θ can be expanded, in particular, to the elements $\tilde{a}(\xi) \in \mathrm{End}\, M_n$, $\xi \in H^*$. By Lemma 1(1), the map $\xi \mapsto \tilde{a}(\xi)$ is continuous, hence, $\theta\tilde{a}$ is also continuous. By Lemma 1(2) there exists a locally regular map $a_1 : G \to \mathrm{End}\, M_n$ such that $\tilde{a}_1 = \theta\tilde{a}$. It is clear that $a_1 = \Theta(a)$, hence, $\Theta(a) \in \mathrm{Cend}_n$.

It follows from (15) that Θ preserves $(\cdot\ {}_g\ \cdot)$, $g \in G$, The continuity of θ^{-1} implies the existence of Θ^{-1} $\square$

If C is a conformal subalgebra of Cend_n then by $W_n(C)$ we denote $\mathrm{Span}_{\Bbbk}\{a(g) \mid a \in C,\ g \in G\} \subseteq W_n$. It is clear that $W_n(C)$ is a subalgebra of W_n.

Definition 5. A conformal subalgebra $C \subseteq \mathrm{Cend}_n$ is said to be *irreducible* if M_n contains no $W_n(C)$-invariant H-submodules except for $\{0\}$ and M_n.

Recall that a left (right) ideal of an algebra is essential (see, e.g., [14]) if it has a nonzero intersection with every nonzero left (right) ideal of the algebra.

Lemma 7. *A conformal subalgebra of the form $\mathcal{F}(H \otimes B_0)$ of Cend_n, where B_0 is a left ideal of $\mathbb{M}_n(H)$, is irreducible if and only if B_0 is essential.*

Proof. Suppose B_0 is an essential left ideal. Then the Goldie theory implies (see, e.g., [14]) that B_0 contains a matrix a which is not a zero divisor in $\mathbb{M}_n(H)$. Let $a = \sum\limits_i f_i \otimes a_i$, $f_i \in H$, $a_i \in \mathbb{M}_n(\Bbbk)$. The element $x = \sum\limits_i \mathcal{F}(1 \otimes f_i \otimes a_i)$ belongs to $C = \mathcal{F}(H \otimes B_0)$, and it follows from (9) that $W_n x(z) \subseteq W_n(C)$ for every $z \in G$. If $0 \neq u = \sum\limits_{k=1}^n h_k \otimes e_k \in M_n$ then

$$x(z)u = \sum_{k=1}^n \sum_i L_z(f_i h_k) \otimes a_i e_k,$$

and for $z = e$ we obtain $x(e)u = au \neq 0$ since a is not a zero divisor. Therefore, $W_n(C)u \supseteq W_n x(e)u = W_n au = M_n$ by Proposition 3, i.e., M_n does not contain even a $W_n(C)$-invariant subspace.

Conversely, suppose $C = \mathcal{F}(H \otimes B_0)$ is an irreducible subalgebra of Cend_n. Since H is a semiprime Noetherian commutative algebra, the matrix algebra $\mathbb{M}_n(H)$ is semiprime left and right Noetherian. In particular, by the Goldie theory $\mathbb{M}_n(H)$ is left and right Ore, and its classical (left) quotient algebra $Q = Q(\mathbb{M}_n(H))$ is semisimple Artinian.

Let R stands for the set of all elements of $\mathbb{M}_n(H)$ invertible in $Q = R^{-1}\mathbb{M}_n(H)$. Assume that B_0 is not essential. Then $R^{-1}B_0$ is a proper left ideal of Q (else B_0 contains an element of R, therefore, B_0 is essential). By the Wedderburn—Artin Theorem, Q is a direct sum of full matrix rings over division algebras. Then $R^{-1}B_0$ has a nonzero right annihilator $\mathrm{ann}_r(R^{-1}B_0)$ in Q. If $0 \neq r^{-1}a \in \mathrm{ann}_r(R^{-1}B_0)$, $r \in R$, $a \in \mathbb{M}_n(H)$, then the right Ore condition implies $r^{-1}a = bs^{-1}$, $s \in R$, $0 \neq b \in \mathbb{M}_n(H)$. It is clear that $B_0 b = 0$, therefore, B_0 has a nonzero right annihilator in $\mathbb{M}_n(H)$.

Denote by M' the H-submodule of M_n generated by all $u \in M_n$ such that $B_0 u = 0$. We have shown $M' \neq 0$, but for every $a \in B_0$, $h \in H$, $z \in G$, $u \in M'$ we have

$$\mathcal{F}(h \otimes a)(z)u = (ha_{(-1)})(z^{-1})a_{(2)}L_z u = h(z^{-1})L_z(au) = 0.$$

Hence, M' is a $W_n(C)$-invariant H-submodule. This is a contradiction with irreducibility of C. ☐

Theorem 4. *Let C be an irreducible subalgebra of Cend_n. Then $C_1 = (1 \otimes H \otimes E)C$ is a left ideal of Cend_n, $C_1 = \mathcal{F}(H \otimes B_0)$, where B_0 is an essential left ideal of $\mathbb{M}_n(H)$.*

Proof. Let $S = W_n(C)$. Then $S_1 = HS$ is also a subalgebra of W_n since $a(g)h = L_g h a(g)$ for $a \in C$, $g \in G$, $h \in H$. It is also clear that $C_1 = (1 \otimes H \otimes E)C$ is a conformal subalgebra of Cend_n, and $S_1 = W_n(C_1)$.

If C is irreducible in the sense of Definition 5 then $S_1 = HS$ acts irreducibly on M_n.

By Theorem 2, S_1 is a dense subalgebra of $\mathrm{End}\, M_n$. By Lemma 5(2), the conformal algebra C_1 can be considered as a topological left S_1-module.

Suppose $\{\alpha_k\}_{k\geq 0}$ is a sequence in S_1 which converges to $\alpha \in W_n$ as $k \to \infty$. Then, by Corollary 1, for every $a \in \mathrm{Cend}_n$ there exists a natural m such that $\alpha_k \cdot a = \alpha \cdot a$ for all $k \geq m$.

Hence, for all $a \in C_1$, $\alpha = b(g)$, $b \in \mathrm{Cend}_n$, $g \in G$, we have $C_1 \ni \alpha_k \cdot a = \alpha \cdot a = (b\ {}_g\ a)$. Therefore, C_1 is a left ideal of Cend_n. Proposition 2 implies $C_1 = \mathcal{F}(H \otimes B_0)$, where B_0 is a left ideal of $\mathbb{M}_n(H)$ which is essential by Lemma 7. $\square$

If $G = \{e\}$ then Theorem 4 turns into the Burnside Theorem. If $G = \mathbb{A}_1$ then essential ideals of $\mathbb{M}_n(H)$ are of the form $\mathbb{M}_n(H)Q$, $\det Q \neq 0$. The following statement is a corollary of Theorem 4.

Theorem 5 ([5, 6])**.** *Let $C \subseteq \mathrm{Cend}_n$ is an irreducible conformal subalgebra (over $\mathbb{A}^1$, $\mathrm{char}\,\Bbbk = 0$). Then either $C = \mathrm{Cend}_{n,Q} := \mathcal{F}(H \otimes \mathbb{M}_n(H)Q)$, $\det Q \neq 0$, or $C = (1 \otimes P^{-1})(H \otimes 1 \otimes \mathbb{M}_n(\Bbbk))(\mathcal{F}(1 \otimes P))$, where $P \in \mathbb{M}_n(H)$, $\det P \in \Bbbk \setminus \{0\}$.*

Theorem 4 is valid not only for connected groups. Let us consider the case of "intermediate complexity" between trivial group and affine line: when G is a finite group. In this case $H = \Bbbk[G] \simeq (\Bbbk G)^*$, where $\Bbbk G$ is the group algebra of G considered as a Hopf algebra in the ordinary way.

Example 3. Let G_1 be a subgroup of a finite group G, $V = G/G_1$. Then $C_{G_1} = \mathrm{Cend}_n^{(G,V)}$ is an irreducible conformal subalgebra of $\mathrm{Cend}_n^{(G,G)}$.

Theorem 6. *Let C be an irreducible conformal subalgebra of $\mathrm{Cend}_n^{(G,G)}$, $|G| < \infty$. Then there exists an automorphism θ of the conformal algebra Cend_n such that $\theta(C) = C_{G_1,\chi}$,*

$$C_{G_1,\chi} = \left\{ \sum_{g\in G} T_g \otimes \sum_{k=1}^{p} \sum_{\alpha\in G_k} T_\alpha \otimes \chi(g,\alpha) a_{g,k} \mid a_{g,k} \in \mathbb{M}_n(\Bbbk) \right\},$$

where G_1 is a subgroup of G, $\{G_1, \dots, G_p\} = \{gG_1 \mid g \in G\}$, and $\chi : G \times G \to \Bbbk^*$ is a function satisfying the following condition: for every $g, h \in G$, $k \in \{1, \dots, p\}$ the value

$$\frac{\chi(g,\gamma)\chi(h, g^{-1}\gamma)}{\chi(gh,\gamma)} \in \Bbbk^*$$

does not depend on the choice of a representative $\gamma \in G_k$.

For example, if $\chi \equiv 1$ then the conformal algebra $\mathrm{Cend}_n^{G,G/G_1}$ is isomorphic to $C_{G_1,\chi}$.

Proof. Sine $H = \mathrm{Span}\{T_g \mid g \in G\}$, $T_g(\gamma) = \delta_{g,\gamma}$, then $T_g T_\gamma = \delta_{g,\gamma} T_g$, and every conformal subalgebra C of $\mathrm{Cend}_n = H \otimes H \otimes \mathbb{M}_n(\Bbbk)$ can be presented as $C = \bigoplus\limits_{g\in G} \Bbbk T_g \otimes S_g$, where $S_g = \{x \in H \otimes \mathbb{M}_n(\Bbbk) \mid T_g \otimes x \in C\}$.

It is easy to see that S_e is a subalgebra of $H \otimes \mathbb{M}_n(\Bbbk)$. Moreover,

$$S_g(L_{g^{-1}} S_h) \subseteq S_{gh} \tag{16}$$

for all $g, h \in G$.

Let us identify $H \otimes \mathbb{M}_n(\Bbbk)$ with $\bigoplus\limits_{\gamma\in G} \mathbb{M}_n(\Bbbk)$ and denote by π_g, $g \in G$, the canonical projections $S_e \to \mathbb{M}_n(\Bbbk)$.

If C is irreducible then Theorem 4 implies

$$\sum_{\gamma\in G} (T_\gamma \otimes 1) S_g = H \otimes \mathbb{M}_n(\Bbbk) \tag{17}$$

for each $g \in G$. Thus, S_e is a subdirect sum of matrix algebras, hence, $S = I_1 \oplus \dots \oplus I_p$, where $I_k \simeq \mathbb{M}_n(\Bbbk)$ are two-sided ideals of S_e. Denote $G_k = \{g \in G \mid \pi_g(I_k) \neq 0\} \subseteq G$, $k = 1, \dots, p$. Since $I_k I_l = 0$ for $k \neq l$, we have $G_k \cap G_l = \emptyset$. It follows from (17) that $G = \bigcup\limits_{1\le k\le p} G_k$. Let us enumerate the sets G_k in such a way that $e \in G_1$.

The maps $\pi_g : I_k \to \mathbb{M}_n(\Bbbk)$, $g \in G_k$, are isomorphisms. Therefore,

$$S_e = \left\{ \sum_{k=1}^{p} \sum_{g\in G_k} T_g \otimes a_k^{\theta_g} \mid a_k \in \mathbb{M}_n(\Bbbk) \right\}, \tag{18}$$

where θ_g, $g \in G$, are automorphisms of the algebra $\mathbb{M}_n(\Bbbk)$.

Relation (17) also implies that for every $g, \gamma \in G$, $a \in \mathbb{M}_n(\Bbbk)$ there exists $x \in S_g$ such that $T_\gamma x := (T_\gamma \otimes 1)x = T_\gamma \otimes a$.

Lemma 8. *The set G_1 is a subgroup of G, and $G/G_1 = \{gG_1 \mid g \in G\} = \{G_k \mid k = 1, \dots, p\}$.*

Proof. Assume there exist $k, m \in \{1, \dots, p\}$, $g \in G$ such that $gG_k \cap G_m \neq \emptyset, G_m$. Consider an arbitrary $x = \sum\limits_{\gamma\in G} T_\gamma \otimes a_\gamma \in S_g$, $a_{\gamma_0} \neq 0$ for some $\gamma_0 \in gG_k \cap G_m$. Then by (16) and (18), we have

$$y_1 = \sum_{\gamma\in gG_k} T_\gamma \otimes a_\gamma \in S_g, \quad y_2 = \sum_{\gamma\in gG_k\cap G_m} T_\gamma \otimes a_\gamma \in S_g.$$

Let us choose $z \in S_{g^{-1}}$ in such a way that $T_{\gamma_0} z = T_{\gamma_0} \otimes E$ and $w = y_2 L_{g^{-1}} z \in S_e$. The element w has the following properties: $T_{\gamma_0} w = a_{\gamma_0} \neq 0$, $T_\gamma w = 0$ for $\gamma \in G_m \setminus gG_k$. This is a contradiction to (18).

Therefore, left multiplication by $g \in G$ permutes the sets $\{G_1, \dots, G_p\}$. □

Lemma 9. 1. *Let $\sigma_{g,\alpha} \in \operatorname{End} \mathbb{M}_n(\Bbbk)$, $g, \alpha \in G$, be a collection of bijective linear transformations. Define an H-linear map σ by the rule*

$$\begin{aligned} &\sigma : \operatorname{Cend}_n \to \operatorname{Cend}_n, \\ &T_g \otimes T_\alpha \otimes a \mapsto T_g \otimes T_\alpha \otimes a^{\sigma_{g,\alpha}}, \quad g, \alpha \in G,\ a \in \mathbb{M}_n(\Bbbk). \end{aligned} \tag{19}$$

The map σ is an automorphism of the conformal algebra Cend_n if and only if $(ab)^{\sigma_{gh,\alpha}} = a^{\sigma_{g,\alpha}} b^{\sigma_{h,g^{-1}\alpha}}$ for all $a, b \in \mathbb{M}_n(\Bbbk)$, $g, h, \alpha \in G$.

2. *For an arbitrary set θ_α, $\alpha \in G$, of automorphisms of the algebra $\mathbb{M}_n(\Bbbk)$ there exists an automorphism σ of the form (19) such that $\sigma_{e,\alpha} = \theta_\alpha$.*

Proof. To prove first statement, it is enough to check the condition $\sigma(x \ {}_g\ y) = \sigma(x) \ {}_g\ \sigma(y)$, $x, y \in \operatorname{Cend}_n$, $g \in G$.

To prove the second one, consider matrices $T_\alpha \in \mathbb{M}_n(\Bbbk)$ such that $T_\alpha^{-1} a T_\alpha = a^{\theta_\alpha}$, and define $a^{\sigma_{g,\alpha}} = a^{\theta_\alpha} E_{g,\alpha}$, where $E_{g,\alpha} = T_\alpha^{-1} T_{g^{-1}\alpha}$, $g, \alpha \in G$, $a \in \mathbb{M}_n(\Bbbk)$. It is easy to check that the maps $\sigma_{g,\alpha} \in \operatorname{End} \mathbb{M}_n(\Bbbk)$ constructed satisfy the conditions of statement 1. □

Therefore, we may suppose that for the conformal subalgebra $C \subseteq \operatorname{Cend}_n$ we have $S_e = \bigoplus_{k=1}^{p} A_k \otimes \mathbb{M}_n(\Bbbk)$, where $A_k = \Bbbk(\sum_{g\in G_k} T_g) \subset H$. For others $g \in G$ the structure of the space S_g can be clarified by means of (16) and (17). Namely,

$$S_g = \left\{ \sum_{k=1}^{p} \sum_{\gamma\in G_k} T_\gamma \otimes a_k^{\sigma_{g,\gamma}} \mid a_k \in \mathbb{M}_n(\Bbbk) \right\},$$

where $\sigma_{g,\gamma}$ are some bijective linear transformations of $\mathbb{M}_n(\Bbbk)$. To be more precise, fix a system of representative $g_k \in G_k$ and assume $\sigma_{g,g_k} = \mathrm{id}$, $k = 1, \dots, p$.

Consider elements of the form

$$x = \sum_{k=1}^{p} \sum_{\alpha\in G_k} T_\alpha \otimes E^{\sigma_{g,\alpha}} \in S_g, \quad y = \sum_{\beta\in G} T_\beta \otimes b \in S_e,$$

where $b \in \mathbb{M}_n(\Bbbk)$ is an arbitrary matrix. Comparing the expressions

$$(T_e \otimes y) \ {}_e\ (T_g \otimes x) = T_g \otimes z, \quad (T_g \otimes x) \ {}_{g^{-1}}\ (T_e \otimes y) = T_g \otimes z',$$

we may conclude that $\pi_{g_k}(z) = \pi_{g_k}(z') = b$ for all $k = 1, \dots, p$. It is easy to derive

$$a^{\sigma_{g,\gamma}} = \chi(g,\gamma)a, \quad \chi : G \times G \to \Bbbk^*,$$

where $\chi(g, g_k) = 1$. Using the relation (16) it is not difficult to obtain the following relations for χ:

$$\chi(g,\alpha)\chi(h, g^{-1}\alpha) = \chi(h, g^{-1}g_k)\chi(gh, \alpha), \quad \alpha \in G_k,\ g, h \in G.$$

Since $C = \sum\limits_{g\in G} T_g \otimes S_g$, we obtain $C = C_{G_1,\chi}$. $\square$

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